



Level set methods for optimization problems involving geometry and constraints II. Optimization over a fixed surface

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ABSTRACT

In this work, we consider an optimization problem described on a surface. The approach is illustrated on the problem of finding a closed curve whose arclength is as small as possible while the area enclosed by the curve is fixed. This problem exemplifies a class of optimization and inverse problems that arise in diverse applications. In our approach, we assume that the surface is given parametrically. A level set formulation for the curve is developed in the surface parameter space. We show how to obtain a formal gradient for the optimization objective, and derive a gradient-type algorithm which minimizes the objective while respecting the constraint. The algorithm is a projection method which has a PDE interpretation. We demonstrate and verify the method in numerical examples.

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1. Introduction

This work represents a continuation of our investigation into optimization problems involving geometry and constraints [12]. In the present study, we are motivated by the need to solve optimization and inverse problems which are described on a surface. The problems are geometric in nature; i.e., we wish to find a set (possibly multiply connected) on the surface which extremizes certain cost functionals. The approach we will present is quite general but we will focus on a specific problem arising in differential geometry.

Consider a smooth fixed surface S included in some bounded open set $\Omega \subset \mathbb{R}^3$. On this surface, we denote a closed curve by Γ (Fig. 1). The arclength of the curve, denoted by $\ell(\Gamma)$, is to be minimized while the area enclosed by the curve is $A(\Gamma)$ is fixed. The optimization problem then is

$$\min_{\text{Area}(\Gamma)=C} \ell(\Gamma).$$

In the planar case, this is a classical problem whose solution is given by the isoperimetric theorem (the unique solution is a circle). On general surfaces the problem is harder and although there have been some recent advances, some open questions remain (see [8] and references therein). The goal of this work is to develop an effective numerical method for solving problems of this type.

In [6], the authors pioneered the use of level set methods for the study of the motion of curves on surfaces. Let S be the surface and assume that it is included in $\Omega \subset \mathbb{R}^3$. In their approach, the surface S is defined as the zero level set function of $\Psi(x) : \Omega \rightarrow \mathbb{R}$ (see [11] for an introduction to level set methods). The curve Γ_t , which moves on S , is a function of time t . The

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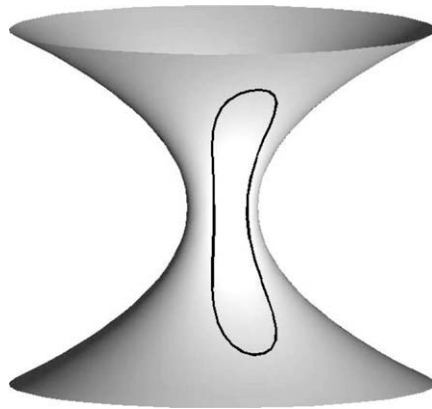


Fig. 1. The optimization problem is to find the curve with the shortest arclength while keeping the area contained by the curve on the surface fixed. This is an instance of an optimization problem involving geometry and constraints.

curve given as the intersection of the zero level sets of $\Psi(x)$ and a time-dependent function $\Phi(x, t) : \Omega \times (0, T) \rightarrow \mathbb{R}$. In their method, both functions $\Phi(x, t)$ and $\Psi(x)$ are given on a Cartesian grid. Differential operations on the surface are computed via projection of operators in \mathbb{R}^3 .

In our work, the surface S is kept fixed. In contrast, we use a parametric representation of S rather than a level set representation. Given that in \mathbb{R}^3 the number of parameters is two, we wish to exploit this fact in our approach. We will still use a level set representation for the curve Γ_t , but it will be given by a function mapping a 2-D domain in parameter space to the reals. This means that our computational method is 2-D, and can be expected to be efficient. Our approach still has the benefits of a level set method. Singularities which could develop, such as merging and splitting, as the curve Γ_t evolves, are easily handled.

The cost in the reduction in computational complexity is an increase in the complexity of the formulas involving differential operations. Nevertheless, this cost may be worthwhile in many situations since the resulting computation can still be carried out in 2-D Cartesian grid, albeit in the surface parameter domain and a narrow-band method can still be implemented.

While the approach we advocate here can be used to address more general optimization problems, the problem we focus on has been treated in [6, Section 11]. What we offer here is an alternative strategy for this class of problem, i.e., curve optimization problem with fixed-area constraint. The advantage of our approach is computational efficiency. Like the approach in [6], ours can also deal with time-dependent supporting surface provided that a parametric representation of the moving surface, S_t , is known.

As in [12], our strategy for the constrained minimization problem is to devise a projected gradient method. The gradient of the cost function is projected onto the tangent space of the constraint. Unlike the method proposed in [12], the projection is done directly to the computed gradient. Moreover, we take the point of view that the iterative method is a gradient flow, thus, allowing for a PDE interpretation, and numerical implementation using methods for solving PDEs.

In order to do the optimization, we need to develop some formulas to calculate such quantities as arclength, area and their variations with respect to the level set function. They will be used to derive an iterative method whereby we start with an initial guess for the curve and proceed to take steps towards minimization by moving the curve.

There has been some notable progress in, as well as new applications of, level set approach for optimal design. Allaire et al. [2] were the first to exploit the level set method for structural optimization. Their method, while quite powerful, was not able to nucleate new holes during the optimization process. To address this dependence on initial condition, Allaire et al. [3] devised a strategy that combines topological derivatives information with level set formulation. Another application comes from photonics [10] where one is interested in designing a nano-structure with prescribed band-gap properties. The question of whether one can devise a topology preserving level set method for optimal design was addressed in [1]. The authors developed penalty terms that can be added to a design objective so that the topology of the initial guess is maintained throughout the optimization process. Finally, we mention a review [4] which demonstrates the use of the level set approach in a variety of inverse and optimal design problems.

The present paper investigates one aspect of optimal design and inverse problem that has not been sufficiently addressed, that of solving a variational problem where the unknown geometry lies on a surface. We assume that the surface is given parametrically and exploit this in devising a computational method. The paper is organized as follows: Section 2 details the general framework for the computational method. We also provide geometrical formulas for arclength and area, and describe curve evolutions which preserve the area. A descent algorithm for curve shortening is presented in Section 3. In Section 4, we derive the equation for the geodesic curvature in terms of the level set function. Additionally, we show that the geodesic curvature is constant on the curve when the velocity for the flow of the curve is zero. Numerical examples are presented in Section 5, where we also validate our computational results. A summarizing discussion is contained in Section 6. For the convenience of the reader, we provide a list of our notation below.

1.1. Notation

The following notation is used throughout the paper:

- $\gamma(r, s) : J^2 \rightarrow \mathbb{R}^3$ is the parameterization of the fixed surface S . In component form $\gamma = (\gamma_1, \gamma_2, \gamma_3)^T$.
- $\nabla = (\partial_r, \partial_s)^T$. The 3-D Cartesian gradient is denoted by ∇_x .
- $\varphi(r, s) = 0$ is the level set function for the curve on S described in the parameter space.
-

$$\nabla\gamma = \begin{pmatrix} \gamma_{1,r} & \gamma_{1,s} \\ \gamma_{2,r} & \gamma_{2,s} \\ \gamma_{3,r} & \gamma_{3,s} \end{pmatrix}, \quad \nabla\gamma^T = \begin{pmatrix} \gamma_{1,r} & \gamma_{2,r} & \gamma_{3,r} \\ \gamma_{1,s} & \gamma_{2,s} & \gamma_{3,s} \end{pmatrix}.$$

•

$$\nabla \times \gamma = \begin{pmatrix} -\gamma_{1,s} & \gamma_{1,r} \\ -\gamma_{2,s} & \gamma_{2,r} \\ -\gamma_{3,s} & \gamma_{3,r} \end{pmatrix}, \quad \nabla \times \gamma^T = \begin{pmatrix} -\gamma_{1,s} & -\gamma_{2,s} & -\gamma_{3,s} \\ \gamma_{1,r} & \gamma_{2,r} & \gamma_{3,r} \end{pmatrix}.$$

- $\nabla\varphi = (\varphi_r, \varphi_s)^T$, and $\nabla \times \varphi = (-\varphi_s, \varphi_r)^T$.
- For 2-vectors u and v ,

$$u \otimes v = \begin{bmatrix} u_1 v_1 & u_1 v_2 \\ u_2 v_1 & u_2 v_2 \end{bmatrix}.$$

- Divergence of a 2×2 matrix is

$$\text{div}A = \begin{bmatrix} A_{11,r} + A_{12,s} \\ A_{21,r} + A_{22,s} \end{bmatrix}.$$

2. Motion of curves on a fixed surface

Since the surface S is fixed, we can choose the following parameterization. Let J be an interval, and $\gamma : J^2 \rightarrow \mathbb{R}^3$ be such that

$$S = \{x | x = \gamma(r, s), \quad (r, s) \in J^2\}.$$

We will view the iterative optimization method as a discretization of a ‘flow’. Therefore, it will be most convenient to consider the problem in the continuous setting. To this end, the curve on the surface is denoted by Γ_t , where the subscript t denotes its dependence on time t . The curve Γ_t is given a level set representation in the parameter domain J^2 . Let $\varphi : J^2 \times (0, T) \rightarrow \mathbb{R}$ such that

$$\Gamma_t = \{x | x = \gamma(r, s), \quad \varphi(r, s, t) = 0\}.$$

We will consider two cases:

- (i) S has a boundary but the curve Γ_t does not touch this boundary. We assume $\varphi > \alpha > 0$ on ∂J^2 .
- (ii) S has no boundary. In that case γ is taken periodic in r and s .

An obvious generalization is the case where S is a truncated cylinder, then γ will be periodic in one direction and φ will be constrained to be positive on the boundary of the parameter space of the other direction. All that follows applies to that case as well.

To move the curve Γ_t , we will evolve the level set function $\varphi(r, s, t)$ according to a transport equation with a given velocity field. To constrain the area enclosed by the curve Γ_t on S , we will need to find a projection for the velocity field. These ideas are discussed in more detail below.

We note that in our formulation the surface is given explicitly whereas the curve on the surface is given implicitly as the zero level set of function $\varphi(r, s)$. This fact requires us to derive formulas for simple quantities such as arclength and area, which are substantially more complicated than those in [6]. We need to work with the parameter variables in order to obtain 2-D equations for the motion of the curve.

2.1. Arclength and surface area

The computation of arclength of Γ_t on the surface S takes a few steps. We introduce a parametric representation of the zero level set $\{(s, r) | \varphi(r, s, t) = 0\}$. Let K be an interval and $\tau \in K$ be a parameter. The map $\beta : \tau \in K \rightarrow J^2$ is such that

$$\varphi(\beta(\tau, t), t) = 0.$$

The curve Γ_t is then $\{x|x = \gamma(\beta(\tau, t))\}$, and it is easy to calculate arclength from this. The length of Γ_t is

$$\ell(\Gamma_t) = \int_K \left| \frac{d}{d\tau} \gamma(\beta(\tau, t)) \right| d\tau = \int_K |\nabla \gamma \beta_{,\tau}| d\tau = \int_K \left| \nabla \gamma \frac{\beta_{,\tau}}{|\beta_{,\tau}|} \right| |\beta_{,\tau}| d\tau.$$

The vector $\beta_{,\tau}/|\beta_{,\tau}|$ is simply the unit tangent on the curve in the parameter domain J^2 . The component $|\beta_{,\tau}| d\tau$ is the infinitesimal arclength on J^2 . We replace both these with their level set function counterparts:

$$\ell(\Gamma_t) = \int_{J^2} \left| \nabla \gamma \frac{\nabla \times \varphi}{|\nabla \varphi|} \right| |\nabla \varphi| \delta(\varphi) dr ds.$$

Here, we have introduced the notation $\nabla \times \varphi = [-\varphi_{,s}, \varphi_{,r}]^T$, and $\delta(\cdot)$ is the Dirac delta function. We will approximate this integral as a limit of an approximate delta function. Letting $\zeta(\cdot)/\varepsilon$ be the approximate delta function, we obtain from above:

$$\ell(\Gamma_t) = \lim_{\varepsilon \rightarrow 0} \int_{J^2} \left| \nabla \gamma \frac{\nabla \times \varphi}{|\nabla \varphi|} \right| |\nabla \varphi| \frac{1}{\varepsilon} \zeta\left(\frac{\varphi}{\varepsilon}\right) dr ds = \lim_{\varepsilon \rightarrow 0} \int_{J^2} |\nabla \gamma \nabla \times \varphi| \frac{1}{\varepsilon} \zeta\left(\frac{\varphi}{\varepsilon}\right) dr ds.$$

Next, letting

$$\nabla \times \gamma = \begin{pmatrix} -\gamma_{1,s} & \gamma_{1,r} \\ -\gamma_{2,s} & \gamma_{2,r} \\ -\gamma_{3,s} & \gamma_{3,r} \end{pmatrix},$$

we denote

$$\ell_\varepsilon(\Gamma_t) = \int_{J^2} |\nabla \gamma \nabla \times \varphi| \frac{1}{\varepsilon} \zeta\left(\frac{\varphi}{\varepsilon}\right) dr ds = \int_{J^2} |\nabla \times \gamma \nabla \varphi| \frac{1}{\varepsilon} \zeta\left(\frac{\varphi}{\varepsilon}\right) dr ds, \tag{1}$$

which represents the approximate arclength of Γ_t .

The area enclosed by Γ_t onto S is a little simpler to calculate. Let $\mathcal{H}(\cdot)$ be the Heaviside function, and $H(\cdot)$ be its approximation, then

$$\text{Area}(\Gamma_t) = \int_{J^2} (1 - \mathcal{H}(\varphi(r, s, t))) |\gamma_{,r} \times \gamma_{,s}| dr ds = \lim_{\varepsilon \rightarrow 0} \int_{J^2} (1 - H\left(\frac{\varphi}{\varepsilon}\right)) |\gamma_{,r} \times \gamma_{,s}| dr ds. \tag{2}$$

(1) and (2) should be compared to those in [6, Section 11].

2.2. Area preserving velocity field

Recall that our goal is to solve the problem:

$$\min_{\Gamma} \ell(\Gamma) \quad \text{subject to } \text{Area}(\Gamma) = C. \tag{3}$$

As we mentioned, our approach will be to obtain a ‘flow’ that reduces the objective while respecting the constraint. The flow deforms the curve by transporting the level set function $\varphi(r, s, t)$. This is done through the equation

$$\varphi_t + w \cdot \nabla \varphi = 0. \tag{4}$$

The velocity field $w(r, s, t)$ will be such that the objective is decreased, but we also need to make certain that the area is preserved. Therefore, we need to determine the condition satisfied by w such that area is preserved during the flow.

We use the approximate area as the surrogate for the area, therefore

$$\text{Area}_\varepsilon(\Gamma_t) := \int_{J^2} (1 - H\left(\frac{\varphi}{\varepsilon}\right)) |\gamma_{,r} \times \gamma_{,s}| dr ds = C.$$

Then differentiating leads to

$$\int_{J^2} -\frac{1}{\varepsilon} \zeta\left(\frac{\varphi}{\varepsilon}\right) \varphi_t |\gamma_{,r} \times \gamma_{,s}| dr ds = 0.$$

Since $\varphi_t + w \cdot \nabla \varphi = 0$, and $-\frac{1}{\varepsilon} \zeta\left(\frac{\varphi}{\varepsilon}\right) \nabla \varphi = \nabla (1 - H\left(\frac{\varphi}{\varepsilon}\right))$, we have, from above,

$$\int_{J^2} [w(r, s, t) |\gamma_{,r} \times \gamma_{,s}|] \cdot \nabla (1 - H\left(\frac{\varphi}{\varepsilon}\right)) dr ds = 0.$$

Thus, for a velocity field to preserve the area, it must satisfy

$$\text{div}(|\gamma_{,r} \times \gamma_{,s}| w) = 0. \tag{5}$$

The next step is to find a velocity field that not only preserves the area, but also reduces the arclength.

3. Descent algorithm

The algorithm we propose is a projected gradient approach. We will describe it in terms of a flow in which the objective function, viewed as energy, is decreased in time. The flow is characterized by a velocity field w for the level set function φ which preserves the area.

Recall that the objective we wish to minimize is the arclength of the curve $\ell_\epsilon(\Gamma_t)$, given in (1). We do this by evolving the curve Γ_t by prescribing velocity w to its level set representation. We posit that the t -derivative of arclength takes the form

$$\frac{d\ell_\epsilon(\Gamma_t)}{dt} = - \int_{\mathbb{R}^2} F(\varphi) \cdot w \, dr \, ds. \tag{6}$$

This can be interpreted ‘physically’ as follows: viewing the arclength $\ell_\epsilon(\Gamma_t)$ as ‘energy’, then its time rate of change is ‘power’, which must take the form of the dot product of ‘force’ $F(\varphi)$ and velocity w . We will show below that this is true, at least formally, by directly calculating the derivative.

We choose a velocity of the form

$$w = \frac{F}{|\gamma_r \times \gamma_s|^2} - \frac{\nabla p}{|\gamma_r \times \gamma_s|}. \tag{7}$$

The choice of the first term is to make a negative contribution to $d\ell_\epsilon(\Gamma_t)/dt$. The second term is for projection to constrain w so that the area inside Γ_t is preserved. The normalizations are chosen for two reasons: first, the choice will be important when we give an interpretation for the ‘force’ F . Second, our particular choice leads to Poisson’s equation for p , which can be solved efficiently using existing packages. In order to determine the term p , we require w to satisfy the constraint (5), which means that

$$\Delta p = \operatorname{div} \left(\frac{F}{|\gamma_r \times \gamma_s|} \right). \tag{8}$$

To see that w so determined leads to a flow that reduces arclength, we substitute F in (7) in (6). We obtain

$$\frac{d\ell_\epsilon(\Gamma_t)}{dt} = - \int_{\mathbb{R}^2} |\gamma_r \times \gamma_s|^2 |w|^2 \, dr \, ds - \int_{\mathbb{R}^2} |\gamma_r \times \gamma_s| w \cdot \nabla p \, dr \, ds.$$

Next, we use the identity

$$\operatorname{div}[(|\gamma_r \times \gamma_s| w) p] = (|\gamma_r \times \gamma_s| w) \cdot \nabla p + \operatorname{div}(|\gamma_r \times \gamma_s| w) p,$$

and integrate by parts to obtain

$$\frac{d\ell_\epsilon(\Gamma_t)}{dt} = - \int_{\mathbb{R}^2} |\gamma_r \times \gamma_s|^2 |w|^2 \, dr \, ds + \int_{\mathbb{R}^2} p \operatorname{div}(|\gamma_r \times \gamma_s| w) \, dr \, ds,$$

using the boundary condition $w|_{\partial\Omega^2} = 0$ (or periodic boundary conditions). By (5) we see that the second term on the right-hand side is zero. Therefore, we have established that

$$\frac{d\ell_\epsilon(\Gamma_t)}{dt} \leq 0$$

and that the length of the curve stops decreasing if, and only if, w vanishes.

Remark 1. We note that the approach we have adopted here can be applied to more general objective functions and constraints. The idea is to determine what velocity w preserves a given constraint, analogous to (5). Next, one would need to determine an appropriate form of the velocity, similar to (7). The approach outlined is nothing more than the projected gradient method – writing the velocity in two parts: one for descent and the other for preserving the constraint (to within linearization). There is nothing special in the method that is specific to the curved surface geometry of the problem. Indeed it is quite general.

3.1. Computation of ‘the force’

Computing forcing term F in terms of φ and γ is a matter of differential calculus and using the transport equation (4). We start by formally differentiating (1):

$$\frac{d\ell_\epsilon(\Gamma_t)}{dt} = \int_{\mathbb{R}^2} \left[|\nabla \times \gamma \nabla \varphi|_t \frac{1}{\epsilon} \zeta \left(\frac{\varphi}{\epsilon} \right) + |\nabla \times \gamma \nabla \varphi| \frac{1}{\epsilon^2} \zeta' \left(\frac{\varphi}{\epsilon} \right) \varphi_t \right] \, dr \, ds. \tag{9}$$

From (4), we have $\frac{1}{\epsilon^2} \zeta' \left(\frac{\varphi}{\epsilon} \right) \varphi_t = -\frac{1}{\epsilon^2} \zeta' \left(\frac{\varphi}{\epsilon} \right) \nabla \varphi \cdot w = -\nabla \left[\frac{1}{\epsilon} \zeta \left(\frac{\varphi}{\epsilon} \right) \right] \cdot w$ so that the second term of the integrand, which we denote by I_2 reads

$$I_2 = -|\nabla \times \gamma \nabla \varphi| \nabla \left[\frac{1}{\varepsilon} \zeta \left(\frac{\varphi}{\varepsilon} \right) \right] \cdot w. \tag{10}$$

We will see later that this term is cancelled by a component of the first term.

We need to get an expression for the first term in the integrand. We start by taking the gradient of the transport equation (4):

$$\nabla \varphi_t + D^2 \varphi w + \nabla w^T \nabla \varphi = 0.$$

Since $\nabla \times \gamma$ is time independent, we can pre-multiply the equation above by it to get

$$(\nabla \times \gamma \nabla \varphi)_t + \nabla \times \gamma D^2 \varphi w + \nabla \times \gamma \nabla w^T \nabla \varphi = 0.$$

Now, taking the scalar product of this equation with $\nabla \times \gamma \nabla \varphi$ gives

$$\frac{1}{2} \frac{\partial}{\partial t} |\nabla \times \gamma \nabla \varphi|^2 + \nabla \varphi^T \nabla \times \gamma^T \nabla \times \gamma D^2 \varphi w + \nabla \varphi^T \nabla \times \gamma^T \nabla \times \gamma \nabla w^T \nabla \varphi = 0.$$

By transposing, and using the identity $(u \otimes v) : A = u^T A v = v^T A^T u$, we arrive at

$$\frac{1}{2} \frac{\partial}{\partial t} |\nabla \times \gamma \nabla \varphi|^2 + (D^2 \varphi \nabla \times \gamma^T \nabla \times \gamma \nabla \varphi) \cdot w + [\nabla \varphi \otimes (\nabla \times \gamma^T \nabla \times \gamma \nabla \varphi)] : \nabla w = 0,$$

which upon division by $|\nabla \times \gamma \nabla \varphi|$ gives

$$|\nabla \times \gamma \nabla \varphi|_t + D^2 \varphi \nabla \times \gamma^T \frac{\nabla \times \gamma \nabla \varphi}{|\nabla \times \gamma \nabla \varphi|} w + \left[\nabla \varphi \otimes \left(\nabla \times \gamma^T \frac{\nabla \times \gamma \nabla \varphi}{|\nabla \times \gamma \nabla \varphi|} \right) \right] : \nabla w = 0. \tag{11}$$

Next we recall two tensor identities involving a matrix A , and vectors a, b , namely

$$\text{div}(a \otimes b) = (\text{div} b) a + (\nabla a) b \text{ and } A : \nabla b = \text{div}(A^T b) - (\text{div} A) \cdot b.$$

Applying the second identity to the third term in (11) gives

$$\begin{aligned} \left(\nabla \varphi \otimes \left(\nabla \times \gamma^T \frac{\nabla \times \gamma \nabla \varphi}{|\nabla \times \gamma \nabla \varphi|} \right) \right) : \nabla w &= \text{div} \left(\left[\nabla \times \gamma^T \frac{\nabla \times \gamma \nabla \varphi}{|\nabla \times \gamma \nabla \varphi|} \otimes \nabla \varphi \right] w \right) - \text{div} \left(\nabla \varphi \otimes \left(\nabla \times \gamma^T \frac{\nabla \times \gamma \nabla \varphi}{|\nabla \times \gamma \nabla \varphi|} \right) \right) \cdot w \\ &= \text{div} \left(\left[\nabla \times \gamma^T \frac{\nabla \times \gamma \nabla \varphi}{|\nabla \times \gamma \nabla \varphi|} \otimes \nabla \varphi \right] w \right) - \text{div} \left(\nabla \times \gamma^T \frac{\nabla \times \gamma \nabla \varphi}{|\nabla \times \gamma \nabla \varphi|} \right) (\nabla \varphi \cdot w) \\ &\quad - D^2 \varphi \nabla \times \gamma^T \frac{\nabla \times \gamma \nabla \varphi}{|\nabla \times \gamma \nabla \varphi|} w, \end{aligned}$$

after applying the first identity. After collecting terms (11) becomes

$$|\nabla \times \gamma \nabla \varphi|_t = \text{div} \left(\nabla \times \gamma^T \frac{\nabla \times \gamma \nabla \varphi}{|\nabla \times \gamma \nabla \varphi|} \right) (\nabla \varphi \cdot w) - \text{div} \left(\left[\nabla \times \gamma^T \frac{\nabla \times \gamma \nabla \varphi}{|\nabla \times \gamma \nabla \varphi|} \otimes \nabla \varphi \right] w \right).$$

We now want to use this expression in (9), thus it will be multiplied by $\frac{1}{\varepsilon} \zeta \left(\frac{\varphi}{\varepsilon} \right)$, and then integrated over J^2 .

Let us first look at what happens to the second term. We multiply it by $\frac{1}{\varepsilon} \zeta \left(\frac{\varphi}{\varepsilon} \right)$ and use the product rule on the divergence to get

$$\begin{aligned} \text{div} \left(\left[\nabla \times \gamma^T \frac{\nabla \times \gamma \nabla \varphi}{|\nabla \times \gamma \nabla \varphi|} \otimes \nabla \varphi \right] w \right) \frac{1}{\varepsilon} \zeta \left(\frac{\varphi}{\varepsilon} \right) &= \text{div} \left(\left[\nabla \times \gamma^T \frac{\nabla \times \gamma \nabla \varphi}{|\nabla \times \gamma \nabla \varphi|} \otimes \nabla \varphi \right] w \frac{1}{\varepsilon} \zeta \left(\frac{\varphi}{\varepsilon} \right) \right) \\ &\quad - \left(\left[\nabla \times \gamma^T \frac{\nabla \times \gamma \nabla \varphi}{|\nabla \times \gamma \nabla \varphi|} \otimes \nabla \varphi \right] w \right) \cdot \nabla \left[\frac{1}{\varepsilon} \zeta \left(\frac{\varphi}{\varepsilon} \right) \right]. \end{aligned} \tag{12}$$

When we integrate the expression on the right-hand side over J^2 , the first term gives zero as long as $\frac{1}{\varepsilon} \zeta \left(\frac{\varphi}{\varepsilon} \right)$ vanishes on ∂J^2 . This was assumed in the case where S has a boundary (case (i)). In that case cancellation of this term occurs as long as the curve is not too close to the boundary of the parameter space. In the case of a closed surface (case (ii)), this term vanishes by periodicity of γ, φ and w . The second term in the right-hand side of (12) becomes

$$\begin{aligned} \left(\nabla \times \gamma^T \frac{\nabla \times \gamma \nabla \varphi}{|\nabla \times \gamma \nabla \varphi|} \cdot \nabla \left[\frac{1}{\varepsilon} \zeta \left(\frac{\varphi}{\varepsilon} \right) \right] \right) (\nabla \varphi \cdot w) &= \left\{ \left[\nabla \varphi \otimes \left(\nabla \times \gamma^T \frac{\nabla \times \gamma \nabla \varphi}{|\nabla \times \gamma \nabla \varphi|} \right) \right] \nabla \left[\frac{1}{\varepsilon} \zeta \left(\frac{\varphi}{\varepsilon} \right) \right] \right\} \cdot w \\ &= \left(\nabla \times \gamma^T \frac{\nabla \times \gamma \nabla \varphi}{|\nabla \times \gamma \nabla \varphi|} \cdot \nabla \left[\frac{1}{\varepsilon} \zeta \left(\frac{\varphi}{\varepsilon} \right) \right] \right) (\nabla \varphi \cdot w) \\ &= \left(\nabla \times \gamma^T \frac{\nabla \times \gamma \nabla \varphi}{|\nabla \times \gamma \nabla \varphi|} \cdot \frac{1}{\varepsilon^2} \zeta' \left(\frac{\varphi}{\varepsilon} \right) \nabla \varphi \right) (\nabla \varphi \cdot w) \\ &= \left(\nabla \varphi^T \nabla \times \gamma^T \frac{\nabla \times \gamma \nabla \varphi}{|\nabla \times \gamma \nabla \varphi|} \frac{1}{\varepsilon^2} \zeta' \left(\frac{\varphi}{\varepsilon} \right) \right) (\nabla \varphi \cdot w) \\ &= |\nabla \times \gamma \nabla \varphi| \frac{1}{\varepsilon^2} \zeta' \left(\frac{\varphi}{\varepsilon} \right) \nabla \varphi \cdot w = |\nabla \times \gamma \nabla \varphi| \nabla \left[\frac{1}{\varepsilon} \zeta \left(\frac{\varphi}{\varepsilon} \right) \right] \cdot w. \end{aligned}$$

This cancels out I_2 in (10). We finally arrive at the expression for the force

$$F(\varphi) = -\operatorname{div}\left(\nabla \times \gamma^T \frac{\nabla \times \gamma \nabla \varphi}{|\nabla \times \gamma \nabla \varphi|}\right) \frac{1}{\varepsilon} \zeta\left(\frac{\varphi}{\varepsilon}\right) \nabla \varphi. \tag{13}$$

Remark 2. Considering the special case where S is a plane, with $\gamma(r, s) = (r, s, 0)^T$ then $\nabla \times \gamma^T \nabla \times \gamma = \mathbb{I}_2$ and $|\nabla \times \gamma \nabla \varphi| = |\nabla \varphi|$ thus we recover the classic formula.

Note however that this holds because the parameter space has a trivial first fundamental form. If we consider another parameterization of the plane, say $\gamma(r, s) = (r^3, s^3, 0)^T$, then we do not recover the classical formula for the curvature. Similarly, the force F is not invariant under a change of parameter space, since it represents an object onto that space.

Remark 3. For a matrix A and a vector v , we have $\operatorname{div}(A^T v) = \operatorname{div} A \cdot v + A : \nabla v$, where $\operatorname{div} A$ is as usual the (size 3) column vector made of divergence of (size 2) row vectors of A . But $\operatorname{div} \nabla \times \gamma = 0$, thus the formula for the force can also be written as

$$F(\varphi) = -\nabla \times \gamma : \nabla \left(\frac{\nabla \times \gamma \nabla \varphi}{|\nabla \times \gamma \nabla \varphi|} \right) \frac{1}{\varepsilon} \zeta\left(\frac{\varphi}{\varepsilon}\right) \nabla \varphi. \tag{14}$$

3.2. Curve moving algorithm

To sum up, the minimization process is done by solving the following system of PDEs:

$$\varphi_t + w \cdot \nabla \varphi = 0, \tag{15}$$

$$w + \frac{1}{|\gamma_{,r} \times \gamma_{,s}|} \nabla p = \frac{1}{|\gamma_{,r} \times \gamma_{,s}|^2} F(\varphi), \tag{16}$$

$$\operatorname{div}(|\gamma_{,r} \times \gamma_{,s}| w) = 0. \tag{17}$$

The evolution terminates when the velocity field w becomes zero.

The divergence-free condition may be implemented by a slightly modified projection method. For example, for the classical Chorin-type projection [5] we perform these steps:

$$\frac{\varphi^{n+1} - \varphi^n}{\delta t} + w^n \cdot \nabla \varphi^n = 0,$$

$$\tilde{w}^{n+1} = \frac{1}{|\gamma_{,r} \times \gamma_{,s}|^2} F(\varphi^{n+1}),$$

$$\Delta p^{n+1} = \operatorname{div}(|\gamma_{,r} \times \gamma_{,s}| \tilde{w}^{n+1}),$$

$$w^{n+1} = \tilde{w}^{n+1} - \frac{1}{|\gamma_{,r} \times \gamma_{,s}|} \nabla p^{n+1}.$$

We may of course use some more advanced time-stepping scheme but this algorithm is presented here for the sake of simplicity. For example we can use $F(\frac{3}{2} \varphi^{n+1} - \frac{1}{2} \varphi^n)$ rather than $F(\varphi^{n+1})$, so that w^{n+1} will approximate the velocity at time $n + \frac{3}{2}$ and the next step in the transport of φ will be more accurate.

4. Geodesic curvature

We will next provide a geometric interpretation of the force F in (14). When the minimization (3) is solved using the algorithm in (15)–(17), the process terminates when the velocity w is zero. Recall from differential geometry that curves which minimize their length under a fixed enclosed area constraint are linked to constant geodesic curvature curves [8]. We will show that the geodesic curvature of the curves becomes constant when the velocity is zero. Further, the geodesic curvature provides a method for verifying numerical calculations.

An intrinsic and simple way to define the geodesic curvature [7] is to use the classical representation of the curve as a level set Φ on S [6]. Then this curvature is defined by

$$\kappa_g = \operatorname{div}_S \frac{\nabla_S \Phi}{|\nabla_S \Phi|}.$$

Note that in this formula, Φ needs only to be defined on S , since the surface operators do not depend on its values outside S . From our formulation, we can easily define a function on S whose level set is the curve, by setting

$$\Phi(\gamma(r, s)) = \varphi(r, s), \quad \forall (r, s) \in J^2. \tag{18}$$

It is therefore possible to express κ_g in terms of γ and φ . We start by taking the gradient (with respect to (r, s)) of (18):

$$\nabla\gamma^T(r, s)\nabla_x\Phi(\gamma(r, s)) = \nabla\varphi(r, s),$$

where ∇_x denotes the usual gradient in \mathbb{R}^3 . We implicitly extended Φ outside S in a smooth but arbitrary way. Now a definition of the surface gradient is

$$\nabla_S\Phi = \nabla\Phi - (\nabla\Phi \cdot n)n,$$

where n is a unit normal to S . Thus, on S ,

$$\nabla\gamma^T(r, s)\nabla_S\Phi(\gamma(r, s)) = \nabla\gamma^T(r, s)\nabla_x\Phi(\gamma(r, s)) - (\nabla\Phi \cdot n)\nabla\gamma^T n = \nabla\varphi(r, s),$$

since $\nabla\gamma^T n = 0$. Hence, $\nabla_S\Phi$ is the vector in \mathbb{R}^3 such that

$$\nabla\gamma^T(r, s)\nabla_S\Phi(\gamma) = \nabla\varphi \quad \text{and} \quad n \cdot \nabla_S\Phi(\gamma) = 0,$$

where $n = \frac{\gamma_r \times \gamma_s}{|\gamma_r \times \gamma_s|}$. This can be written as

$$A\nabla_S\Phi(\gamma) = \begin{pmatrix} \nabla\varphi \\ 0 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} \gamma_r^T \\ \gamma_s^T \\ (\gamma_r \times \gamma_s)^T \end{pmatrix}.$$

The following holds for vectors a and b :

$$\begin{pmatrix} a^T \\ b^T \\ (a \times b)^T \end{pmatrix}^{-1} = \frac{1}{|a \times b|^2} \begin{pmatrix} b \times (a \times b) & -a \times (a \times b) & a \times b \end{pmatrix}.$$

Applying this to calculate the inverse of A , we obtain

$$\begin{aligned} \nabla_S\Phi(\gamma) &= \frac{1}{|\gamma_r \times \gamma_s|} \begin{pmatrix} \gamma_s \times n & -\gamma_r \times n & n \end{pmatrix} \begin{pmatrix} \nabla\varphi \\ 0 \end{pmatrix} \\ &= \frac{1}{|\gamma_r \times \gamma_s|} [\varphi_r(\gamma_s \times n) - \varphi_s(\gamma_r \times n)]. \end{aligned}$$

Using the triple cross-product formula, we have

$$\begin{aligned} \gamma_s \times n &= \frac{1}{|\gamma_r \times \gamma_s|} [|\gamma_s|^2 \gamma_r - (\gamma_r \cdot \gamma_s) \gamma_s], \\ \gamma_r \times n &= \frac{1}{|\gamma_r \times \gamma_s|} [(\gamma_r \cdot \gamma_s) \gamma_r - |\gamma_s|^2 \gamma_s]. \end{aligned}$$

Thus,

$$\begin{aligned} \nabla_S\Phi(\gamma) &= \frac{1}{|\gamma_r \times \gamma_s|^2} [(|\gamma_s|^2 \varphi_r - (\gamma_r \cdot \gamma_s) \varphi_s) \gamma_r + (|\gamma_r|^2 \varphi_s - (\gamma_r \cdot \gamma_s) \varphi_r) \gamma_s] \\ &= \frac{1}{|\gamma_r \times \gamma_s|^2} \nabla\gamma \begin{pmatrix} (|\gamma_s|^2 \varphi_r - (\gamma_r \cdot \gamma_s) \varphi_s) \\ -(\gamma_r \cdot \gamma_s) \varphi_r + |\gamma_r|^2 \varphi_s \end{pmatrix} = \frac{1}{|\gamma_r \times \gamma_s|^2} \nabla\gamma \begin{pmatrix} |\gamma_s|^2 & -\gamma_r \cdot \gamma_s \\ -\gamma_r \cdot \gamma_s & |\gamma_r|^2 \end{pmatrix} \nabla\varphi \\ &= \nabla\gamma \begin{pmatrix} |\gamma_r|^2 & \gamma_r \cdot \gamma_s \\ \gamma_r \cdot \gamma_s & |\gamma_s|^2 \end{pmatrix}^{-1} \nabla\varphi. \end{aligned}$$

Hence, the surface gradient is expressed in term of φ and γ by

$$\nabla_S\Phi(\gamma) = \nabla\gamma(\nabla\gamma^T\nabla\gamma)^{-1}\nabla\varphi. \tag{19}$$

Note that this formula should not depend on the parameterization, since the surface gradient is intrinsic. Let us check that this is indeed the case by considering a diffeomorphism $\theta : J^2 \rightarrow J^2$ which defines a new parameterization and level set such that $\gamma = \tilde{\gamma}(\theta)$ and $\varphi = \tilde{\varphi}(\theta)$. Then plugging these relations in the expression for surface gradient leads to

$$\nabla_S\Phi(\gamma) = \nabla\tilde{\gamma}\nabla\theta(\nabla\theta^T\nabla\tilde{\gamma}^T\nabla\tilde{\gamma}\nabla\theta)^{-1}\nabla\theta^T\nabla\tilde{\varphi} = \nabla\tilde{\gamma}(\nabla\tilde{\gamma}^T\nabla\tilde{\gamma})^{-1}\nabla\tilde{\varphi}.$$

In order to compute the geodesic curvature we need now to write the divergence operator. The expression for surface gradient reads component-wise:

$$\partial_S^i \Phi = \mathcal{A}_{ix} \varphi_{u_x}, \quad \mathcal{A} = \nabla\gamma(\nabla\gamma^T\nabla\gamma)^{-1} \in \mathbb{R}^{3 \times 2},$$

where $(u_1, u_2) = (r, s)$ and the summation over repeated indices has been used. Thus, for a velocity field V defined on S by $V(\gamma(r, s)) = v(r, s)$ with v defined from J^2 to \mathbb{R}^3 , we get

$$\operatorname{div}_S V(\gamma) = \partial_S^i V_i(\gamma) = \mathcal{A}_{ix} v_{i, u_x} = \mathcal{A} : \nabla v,$$

which leads to the following formula for geodesic curvature:

$$\kappa_g = \mathcal{A} : \nabla \left(\frac{\mathcal{A} \nabla \varphi}{|\mathcal{A} \nabla \varphi|} \right), \quad \mathcal{A} = \nabla \gamma (\nabla \gamma^T \nabla \gamma)^{-1}. \tag{20}$$

We would like to connect this expression to the force in (13). We know that the minimizer of our optimization problem is somehow related to curves such that $\kappa_g = \text{constant}$ [8]. If such a relation is available, we would be able to state what the force satisfies at termination of the evolution. Before doing this let us make a few observations.

Remark 4. The following identities hold:

$$\begin{aligned} (\nabla \gamma^T \nabla \gamma)^{-1} &= \frac{1}{|\gamma_r \times \gamma_s|^2} \nabla \times \gamma^T \nabla \times \gamma, \\ \mathcal{A}^T \mathcal{A} &= (\nabla \gamma^T \nabla \gamma)^{-T} \nabla \gamma^T \nabla \gamma (\nabla \gamma^T \nabla \gamma)^{-1} = (\nabla \gamma^T \nabla \gamma)^{-1}. \end{aligned}$$

Remark 5. Using the above identities, we have

$$|\mathcal{A} \nabla \varphi|^2 = \langle \mathcal{A} \nabla \varphi, \mathcal{A} \nabla \varphi \rangle = \langle \nabla \varphi, \mathcal{A}^T \mathcal{A} \nabla \varphi \rangle = \langle \nabla \varphi, (\nabla \gamma^T \nabla \gamma)^{-1} \nabla \varphi \rangle = \frac{1}{|\gamma_r \times \gamma_s|^2} \langle \nabla \varphi, \nabla \times \gamma^T \nabla \times \gamma \nabla \varphi \rangle,$$

thus

$$|\mathcal{A} \nabla \varphi| = \frac{|\nabla \times \gamma \nabla \varphi|}{|\gamma_r \times \gamma_s|}.$$

We next calculate $|\gamma_r \times \gamma_s| \kappa_g$:

$$|\gamma_r \times \gamma_s| \kappa_g = (|\gamma_r \times \gamma_s| \mathcal{A}) : \nabla \left(\frac{\mathcal{A} \nabla \varphi}{|\mathcal{A} \nabla \varphi|} \right) = \operatorname{div} \left(\frac{|\gamma_r \times \gamma_s| \mathcal{A}^T \mathcal{A} \nabla \varphi}{|\mathcal{A} \nabla \varphi|} \right) - \operatorname{div}(|\gamma_r \times \gamma_s| \mathcal{A}) \cdot \frac{\mathcal{A} \nabla \varphi}{|\mathcal{A} \nabla \varphi|}.$$

We use the identities in Remark 3 in the first term on the right-hand side, and rewrite the left-hand side to get

$$|\gamma_r \times \gamma_s| \kappa_g = \operatorname{div} \left(\nabla \times \gamma^T \frac{\nabla \times \gamma \nabla \varphi}{|\nabla \times \gamma \nabla \varphi|} \right) - \mathcal{A}^T \operatorname{div}(|\gamma_r \times \gamma_s| \mathcal{A}) \cdot \frac{\nabla \varphi}{|\mathcal{A} \nabla \varphi|}.$$

We will show below that $\mathcal{A}^T \operatorname{div}(|\gamma_r \times \gamma_s| \mathcal{A}) = 0$ so that we have the following formula for geodesic curvature:

$$|\gamma_r \times \gamma_s| \kappa_g = \operatorname{div} \left(\nabla \times \gamma^T \frac{\nabla \times \gamma \nabla \varphi}{|\nabla \times \gamma \nabla \varphi|} \right). \tag{21}$$

We pause to examine (21) and connect it with the formula for the force in (13). It can be seen that

$$F(\varphi) = -|\gamma_r \times \gamma_s| \kappa_g \frac{1}{\varepsilon} \zeta \left(\frac{\varphi}{\varepsilon} \right) \nabla \varphi. \tag{22}$$

We showed at the beginning of Section 3 that the curve length stops decreasing when the velocity w is zero in (16). Let us show that we get the expected minimizer. As $w = 0$, from (16) there holds

$$\frac{1}{|\gamma_r \times \gamma_s|} \nabla p = \frac{1}{|\gamma_r \times \gamma_s|^2} F(\varphi).$$

Remark 6. One could be surprised that the force does not vanish at equilibrium. This is due to the fact that the curve is still willing to shorten its length, but is prevented from doing so by the area constraint. Thus, this generates a gradient-like force (corrected by the metric).

Then from (22),

$$\nabla p = -\kappa_g \frac{1}{\varepsilon} \zeta \left(\frac{\varphi}{\varepsilon} \right) \nabla \varphi = -\kappa_g \nabla \left[Z \left(\frac{\varphi}{\varepsilon} \right) \right], \tag{23}$$

where $Z'(r) = \zeta(r)$. Intuitively for this to hold κ_g must be constant in the direction orthogonal to the nablas. This shows that $-\kappa_g$ has to be constant along level sets of φ in a neighborhood of $\varphi = 0$. To show this more rigorously, we can use the curl of a 2-D vector field which is the scalar defined by $\operatorname{curl} v := v_{2, x_1} - v_{1, x_2}$ and verifies for a scalar function f and a velocity field v , $\operatorname{curl}(fv) = f \operatorname{curl} v + \nabla \times f \cdot u = f \operatorname{curl} v - \nabla f \cdot u^\perp$ where u^\perp is orthogonal to u . Thus, taking the curl of (23) above gives

$$0 = -\nabla \kappa_g \cdot \nabla \times Z \left(\frac{\varphi}{\varepsilon} \right) = -\nabla \kappa_g \cdot \nabla \times \varphi \frac{1}{\varepsilon} \zeta \left(\frac{\varphi}{\varepsilon} \right),$$

which just says that κ_g is constant along the tangent to the level sets of φ when $\zeta > 0$, i.e., in a neighborhood of $\varphi = 0$. This proves that our minimizing curves have, as expected, constant geodesic curvature [8]. We thus proved.

Proposition 1. Algorithms (15)–(17) make the length of curve defined as the image of the zero level set of φ by γ diminishing in time. If the length reaches an equilibrium, then the corresponding curve has constant geodesic curvature.

Turning back to show that $\mathcal{A}^T \text{div}(|\gamma_r \times \gamma_s| \mathcal{A}) = 0$, we note that

$$\mathcal{A}^T = (\nabla \gamma^T \nabla \gamma)^{-T} \nabla \gamma^T,$$

so that the requirement is equivalent to

$$\nabla \gamma^T \text{div}(|\gamma_r \times \gamma_s| \mathcal{A}) = 0. \tag{24}$$

Demonstrating (24) is through brute-force calculation. We had hoped to find a clever known fact from geometry to help us but we were unable to do so.

We begin by calculating $(\nabla \gamma^T \nabla \gamma)$:

$$(\nabla \gamma^T \nabla \gamma) = \begin{pmatrix} \gamma_r \cdot \gamma_r & \gamma_r \cdot \gamma_s \\ \gamma_r \cdot \gamma_s & \gamma_s \cdot \gamma_s \end{pmatrix}.$$

Computing the inverse of this 2×2 matrix and using it in the definition of \mathcal{A} , we get

$$\mathcal{A} = \frac{1}{\det(\nabla \gamma^T \nabla \gamma)} \left((\gamma_s \cdot \gamma_s) \gamma_r - (\gamma_r \cdot \gamma_s) \gamma_s \quad -(\gamma_r \cdot \gamma_s) \gamma_r + (\gamma_r \cdot \gamma_r) \gamma_s \right).$$

Using the fact that $|\gamma_r \times \gamma_s| = \sqrt{\det(\nabla \gamma^T \nabla \gamma)}$, we obtain

$$\text{div}(|\gamma_r \times \gamma_s| \mathcal{A}) = \left(\frac{(\gamma_s \cdot \gamma_s) \gamma_r - (\gamma_r \cdot \gamma_s) \gamma_s}{\sqrt{\det(\nabla \gamma^T \nabla \gamma)}} \right)_r + \left(\frac{-(\gamma_r \cdot \gamma_s) \gamma_r + (\gamma_r \cdot \gamma_r) \gamma_s}{\sqrt{\det(\nabla \gamma^T \nabla \gamma)}} \right)_s.$$

Calculations showing that

$$\gamma_r \cdot \left(\frac{(\gamma_s \cdot \gamma_s) \gamma_r - (\gamma_r \cdot \gamma_s) \gamma_s}{\sqrt{\det(\nabla \gamma^T \nabla \gamma)}} \right)_r + \gamma_r \cdot \left(\frac{-(\gamma_r \cdot \gamma_s) \gamma_r + (\gamma_r \cdot \gamma_r) \gamma_s}{\sqrt{\det(\nabla \gamma^T \nabla \gamma)}} \right)_s = 0$$

needed to prove the first of (24) are omitted as they are too tedious. A similar result involving γ_s also holds. Thus, we can conclude that the formula for geodesic curvature in (21) is indeed correct.

5. Numerical examples

In the context of moving a curve with given enclosed surface, our projection algorithm has a clear advantage over other algorithms that use a penalty term to enforce the fixed area constraint. Our implementation uses a MAC grid which ensures accurate divergence-free condition [5], even in the case where γ is not identity. Indeed we project the velocity field $|\gamma_r \times \gamma_s| \tilde{w}(r, s) = |\gamma_r \times \gamma_s| (\tilde{u}(r, s), \tilde{v}(r, s))$ onto the space of divergence-free vector fields by solving a Poisson equation in (r, s) plane. Using our MAC grid this amounts to solve for p (for simplicity we take the grid spacing $dr = ds = h$):

$$\frac{p_{i+1,j} + p_{i-1,j} + p_{i,j+1} + p_{i,j-1} - 4p_{i,j}}{h^2} = \frac{1}{h} \left[|\gamma_r \times \gamma_s|_{i+\frac{1}{2},j} \tilde{u}_{i+\frac{1}{2},j} - |\gamma_r \times \gamma_s|_{i-\frac{1}{2},j} \tilde{u}_{i-\frac{1}{2},j} \right] + \frac{1}{h} \left[|\gamma_r \times \gamma_s|_{i,j+\frac{1}{2}} \tilde{v}_{i,j+\frac{1}{2}} - |\gamma_r \times \gamma_s|_{i,j-\frac{1}{2}} \tilde{v}_{i,j-\frac{1}{2}} \right]$$

and then to update the velocity by

$$u_{i+\frac{1}{2},j} = \tilde{u}_{i+\frac{1}{2},j} - \frac{1}{|\gamma_r \times \gamma_s|_{i+\frac{1}{2},j}} \frac{p_{i+1,j} - p_{i,j}}{h},$$

$$v_{i,j+\frac{1}{2}} = \tilde{v}_{i,j+\frac{1}{2}} - \frac{1}{|\gamma_r \times \gamma_s|_{i,j+\frac{1}{2}}} \frac{p_{i,j+1} - p_{i,j}}{h}.$$

Performing these two steps produce a velocity field w which is divergence-free at the discrete level, since from the above equations there holds

$$\frac{1}{h} \left[|\gamma_r \times \gamma_s|_{i+\frac{1}{2},j} u_{i+\frac{1}{2},j} - |\gamma_r \times \gamma_s|_{i-\frac{1}{2},j} u_{i-\frac{1}{2},j} \right] + \frac{1}{h} \left[|\gamma_r \times \gamma_s|_{i,j+\frac{1}{2}} v_{i,j+\frac{1}{2}} - |\gamma_r \times \gamma_s|_{i,j-\frac{1}{2}} v_{i,j-\frac{1}{2}} \right] = 0.$$

Thus, the surface area constraint is not penalized but enforced. Surface area loss from initialization to stationary state in the case of an ellipse on a cylinder relaxing to a circle is about 2% for a 64×64 grid, under 1% (0.66%) for a 128×128 grid and

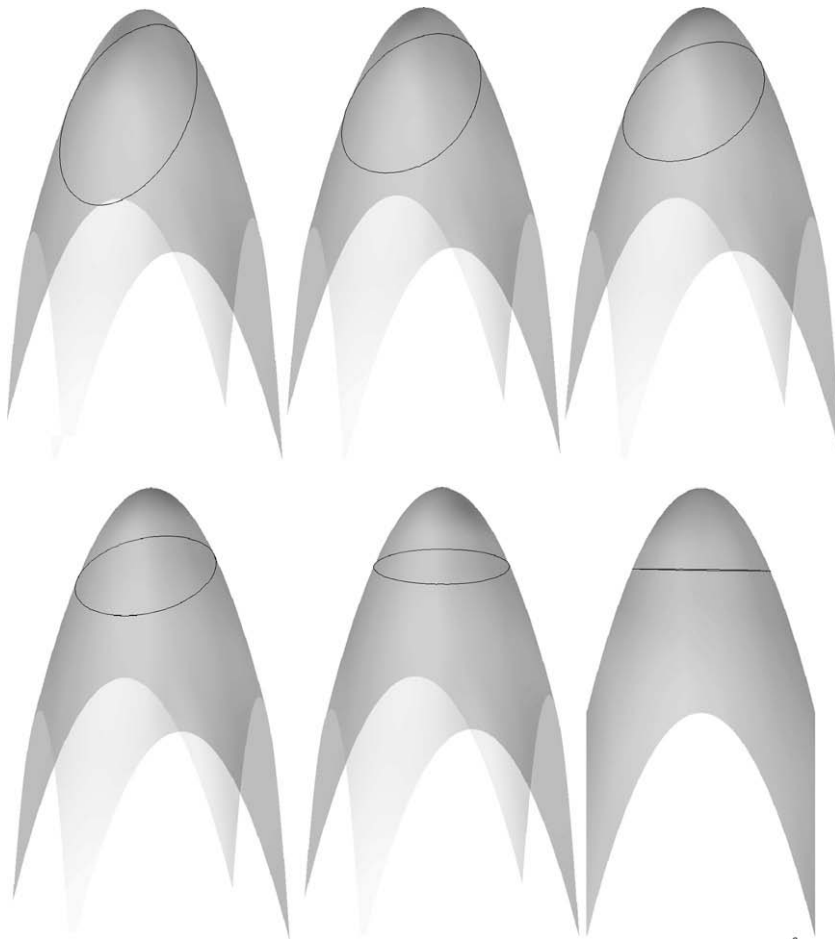


Fig. 2. Minimization of curve length at prescribed enclosed surface area, on a paraboloid. Convergence toward the horizontal circle. Last picture shows a non-perspective plot of the final state.

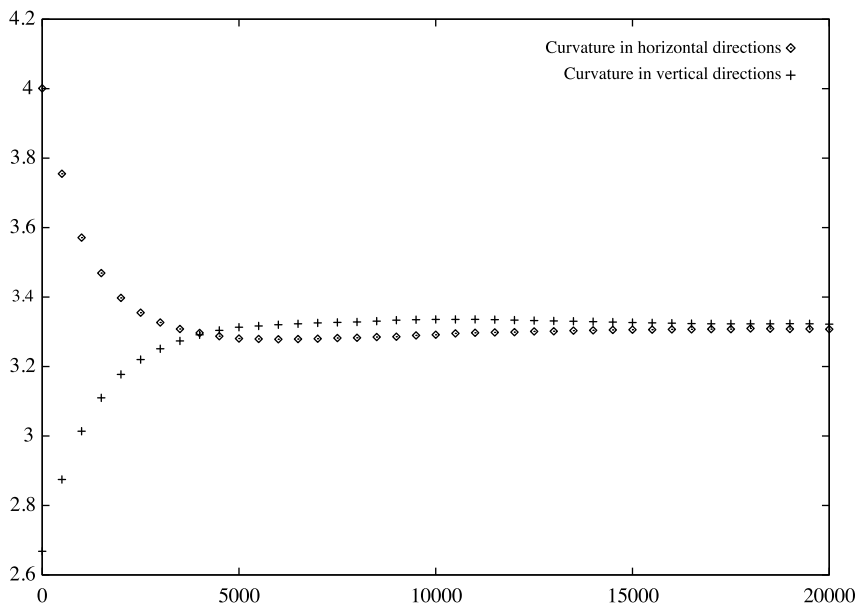


Fig. 3. Convergence to a constant curvature curve. Horizontal and vertical curvatures on the evolving curve are plotted.

0.06% for a 256×256 grid (see Fig. 4). Moreover, as the Poisson equation associated to the projection method lies on the rectangular parametric space, fast FFT solvers (e.g. FISHPACK [13]) may be used, leading to very small computational costs.

The boundary conditions are of Dirichlet type in case of non-closed supporting surfaces, and periodic in one direction in the case of surfaces of revolution. Note however that our algorithm as presented above in its native form requires a regular parametrical representation of the supporting surface in the neighborhood of the moving curve. This fact rules out, for example, the case where the supporting surface is a closed sphere, unless if the curve remains away from the singular poles. In this last case the algorithm works since the force is localized around the curve. In order to deal with a sphere without any *a priori* extra information on the curve motion, we have to adapt the algorithm to handle parametrical patches. This work is under development.

Numerically, we found different ways to compute the force. While each give overall the same evolution, some are more stable than others. In this respect it is worth noticing that our problem has no built-in diffusion, which could regularize some numerical oscillations. For that reason we use WENO [9] schemes to solve the advection equation (15) and to compute gra-

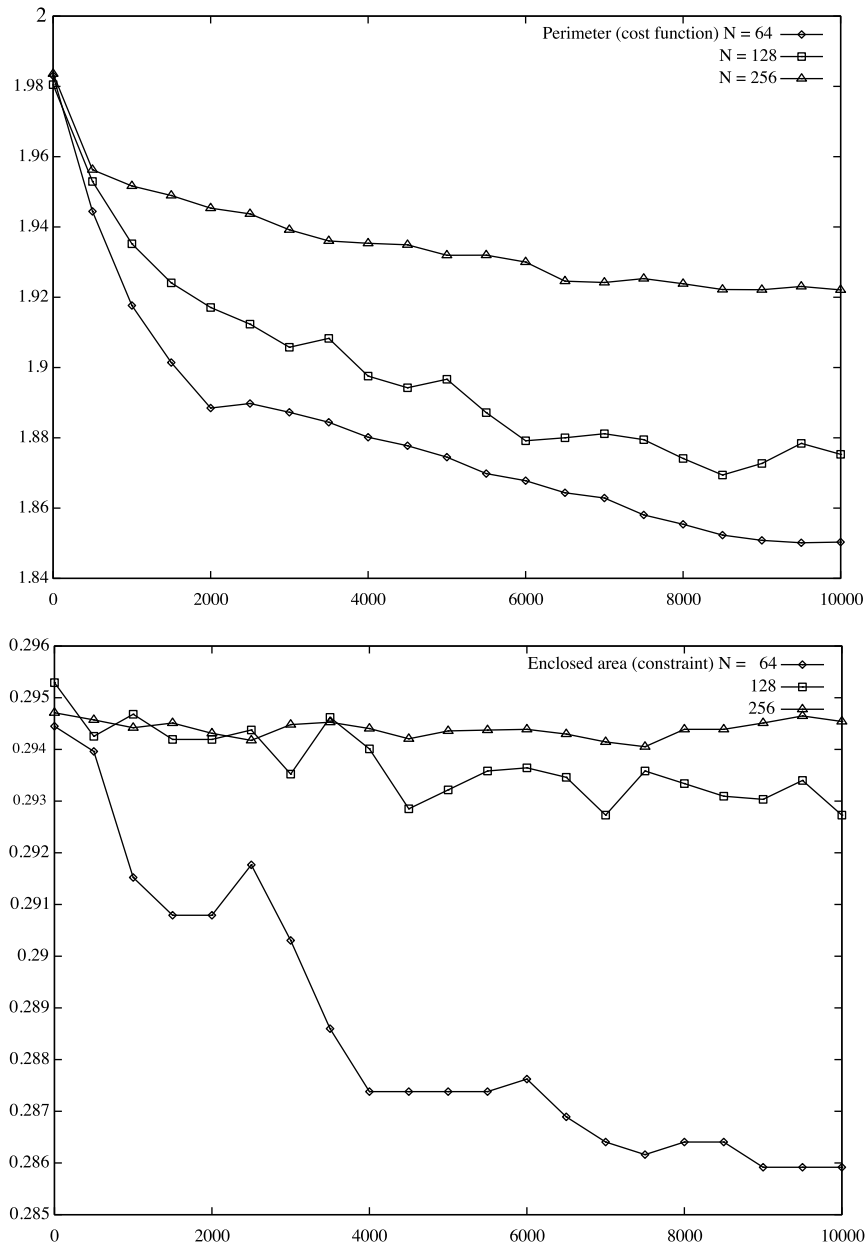


Fig. 4. Cost function (top) and constraint (bottom) for 64, 128 and 256 grid points.

dients of the level set function. We found that the form (14) leads to a more stable evolution than the divergence form (13). An even more stable form could be found by using the identity

$$A : \nabla \left(\frac{A \nabla \varphi}{|A \nabla \varphi|} \right) = \frac{1}{|A \nabla \varphi|} A : \left(\mathbb{I} - \frac{A \nabla \varphi \otimes A \nabla \varphi}{|A \nabla \varphi|^2} \right) \nabla (A \nabla \varphi),$$

applied with $A = \nabla \times \gamma$ to Eq. (14). Note that in the classical computation of curvature on a plane, one uses an expanded form involving second order partial derivatives of φ . This might be done here too (by expanding the last gradient term), but would lead to a huge formula. This intermediate formula showed good stability behavior while remaining relatively easy to implement.

5.1. Paraboloid supporting surface

We first demonstrate our minimization algorithm on a simple problem of finding the shortest closed curve on a paraboloid. The minimizer is known to be a circle. In Fig. 2, we show the evolution of the minimization starting with an ellipsoid on the paraboloid. As can be seen the flow ends with a horizontal circle.

5.2. Cylindrical supporting surface

In the next calculation, we demonstrate two properties of our minimization algorithm – area conservation and correctness of solution. To demonstrate the latter, we consider the minimizing the length of a closed curve on a circular cylinder, where the initial guess for the curve is an ellipse. There are two possible outcomes, depending on the radius of the cylinder [8]. The simplest case, without topological changes, is when the cylinder has a radius large enough so that the minimizer is a circle (see below for the other case). With the cylinder oriented vertically, we evaluated the geodesic curvature on the curve. The value of the geodesic curvature is sampled at two points, corresponding to the vertical and horizontal (with respect to the orientation of the cylinder) curvatures. In Fig. 3, we plot these horizontal and vertical curvatures as a function of evolution. Both converge towards a common value which is the curvature of the minimizing circle. The figure also demonstrates the claimed area conservation. If area is lost during evolution, the radius of the final circle will continue to decrease, thus increasing the curvature. As can be seen in the figure, the curvatures' asymptotic behavior is horizontal. We conclude that area conservation property holds reasonably even in the rough grid 64×64 used. In Fig. 4, we plot the length (cost function) and area (constraint) for $n = 64, 128, 256$. The area loss is respectively 2%, 0.66% and 0.06%. Note that area loss for low resolution leads to a smaller cost function, since the perimeter is also reduced. The area of the minimizer is theoretically $2\pi\sqrt{0.25 \times 0.375} \approx 1.924$. In the next example, the supporting surface is a cylinder of radius $a = 1$. An ellipse in the parametric space is chosen as initialization, which gives the curve drawn on the left-most picture in Fig. 5. This curve is wrapped around the cylinder: the top and bottom loops are running on the back part of the surface while the thinnest part of domain enclosed by the curve is drawn on the front. Computations are made on a 128×128 grid. Due to the fact that the area enclosed by the curve is greater than $4\pi a^2$, the minimizing curve is known to be made of two circles [8], a fact that our computations recover. We plotted in Fig. 6 some steps of this minimization process drawn in the parametric space (r, s) . In Fig. 7, we plotted the evolution of length and area while minimization occurs. The steep variation in length corresponds to the topological change.

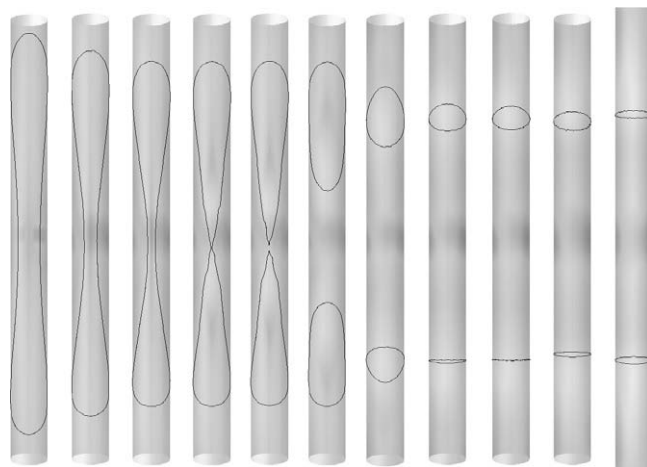


Fig. 5. Minimization of curve length leading to a topological change. Last picture on the right is without perspective to demonstrate symmetry property.

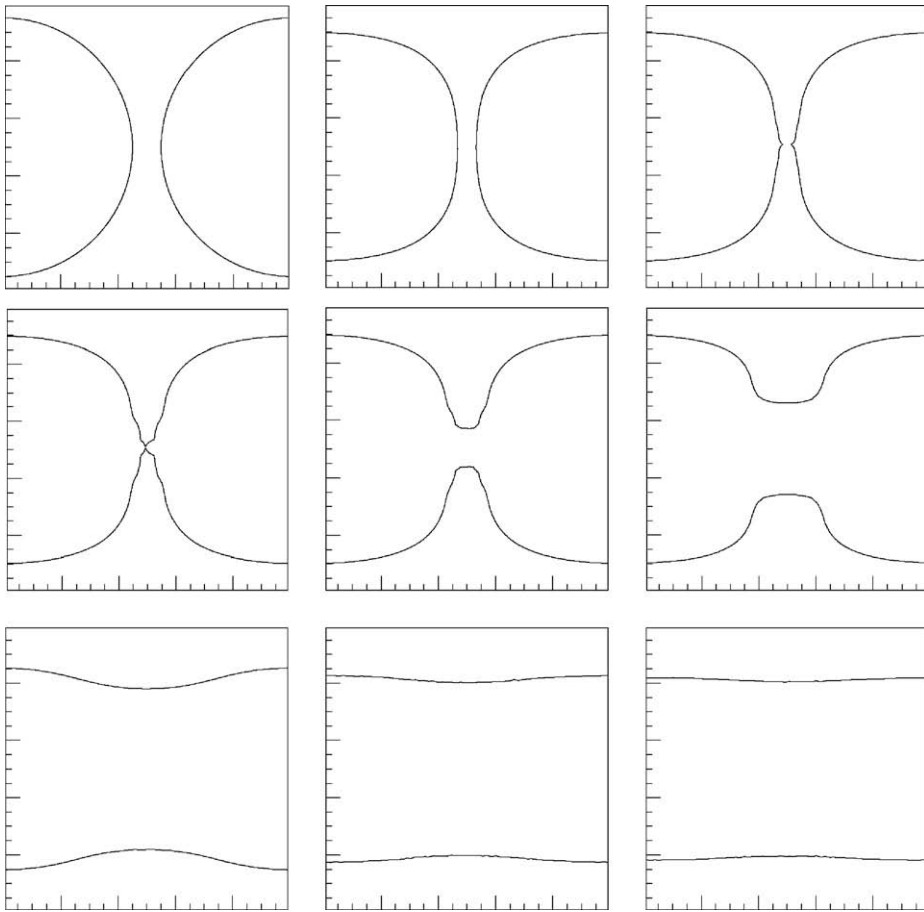


Fig. 6. Same minimization as Fig. 5. Selected iteration steps in the parametric space.

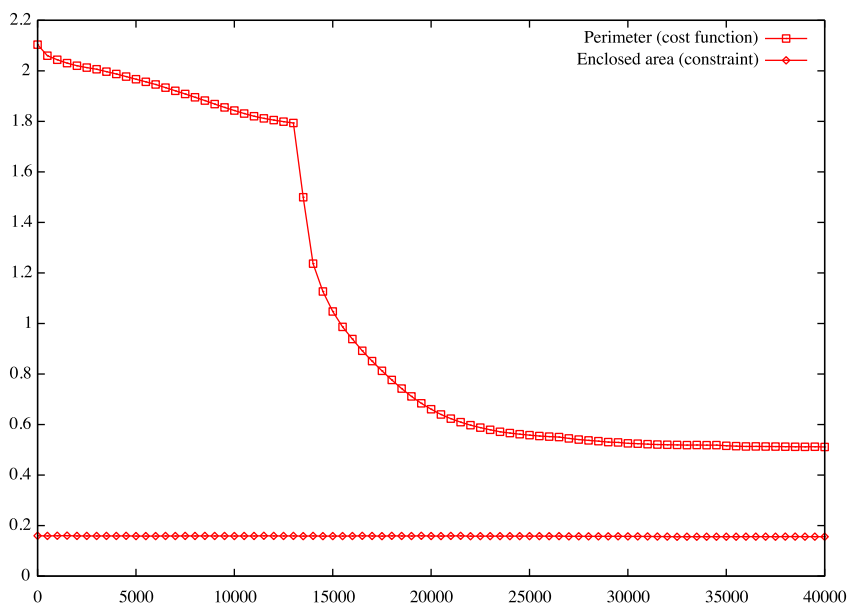


Fig. 7. Length and area variation in the cylinder case with topological change.

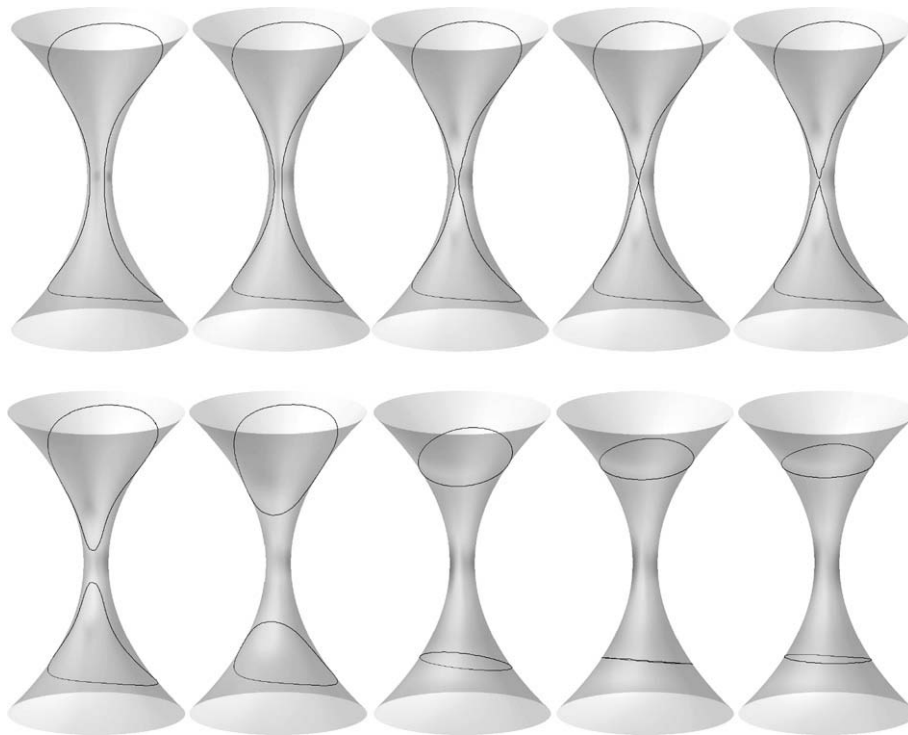


Fig. 8. Minimization of curve length at prescribed enclosed surface area, in the case of a non-flat surface.

5.3. Hyperboloid supporting surface

In the preceding example the metric was flat. To illustrate the fact that our method works for an arbitrary supporting surface, we consider the hyperboloid shape of Fig. 8 and perform the same kind of minimization, starting from an ellipse in the parametrical space. This leads also to a minimizer which is made of two closed circles.

6. Discussion

In this work, we have considered a geometrical optimization problem on a fixed curved surface. As a model, we considered the isoparametric problem of finding a curve of least length with a given area. The method we propose uses a level set function to represent the unknown geometry. The level set function is defined in the 2-D parameter space. Thus, the computation takes place in two dimensions, leading to a very efficient method. The level set function's evolution is governed by a constrained gradient flow which reduces the arclength of the curve. The velocity field is calculated by a projection method. Thus, the curve moves in such a way that the enclosed area remains constant. The approach we propose is a framework for inverse and optimization problems on curved surfaces. In a future work, we will apply the strategy on an inverse problem involving geometry on curved surfaces.

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References

- [1] O. Alexandrov, F. Santosa, A topology-preserving level set method for shape optimization, *J. Comp. Phys.* 204 (2005) 121–130.
- [2] G. Allaire, F. Jouve, A.-M. Toader, Structural optimization using sensitivity analysis and a level-set method, *J. Comp. Phys.* 194 (2004) 363–393.
- [3] G. Allaire, F. De Gournay, F. Jouve, A.-M. Toader, Structural optimization using topological and shape sensitivity via a level set method, *Control Cybernet* 34 (2005) 59–80.
- [4] M. Burger, S. Osher, A survey on level set methods for inverse problems and optimal design, *Eur. J. Appl. Math.* 16 (2005) 263–301.
- [5] D. Brown, R. Cortez, M. Minion, Accurate projection methods for the incompressible Navier–Stokes equations, *J. Comp. Phys.* 168 (2001) 464–499.

- [6] L.-T. Cheng, P. Burchard, B. Merriman, S.J. Osher, Motion of curves constrained on surfaces using a level set approach, *J. Comp. Phys.* 175 (2002) 604–644.
- [7] M. Do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice Hall, New Jersey, 1976.
- [8] H. Howards, M. Hutchings, F. Morgan, The isoperimetric problem on surfaces, *Am. Math. Month.* 106 (5) (1999) 430–439.
- [9] G. Jiang, C.-W. Shu, Efficient implementation of weighted ENO schemes, *J. Comp. Phys.* 126 (1996) 202–228.
- [10] C.-Y. Kao, S. Osher, E. Yablonovitch, Maximizing band gaps in two-dimensional photonic crystals by using level set methods, *Appl. Phys. B: Lasers Opt.* 81 (2005) 235–244.
- [11] S. Osher, R. Fedkiw, *Level Set Methods and Dynamic Implicit Surfaces*, Springer, New York, 2003.
- [12] S.J. Osher, F. Santosa, Level set methods for optimization problems involving geometry and constraints I. Frequencies of a two-density inhomogeneous drum, *J. Comp. Phys.* 171 (2001) 72–288.
- [13] P. Swartztrauber, R. Sweet, J. Adams, FISHPACK: Efficient FORTRAN subprograms for the solution of elliptic partial differential equations, UCAR Technical Report, 1999, <www.cisl.ucar.edu>.